The problem:

Find a formula for the n^{th} non-square number.

The predefined resources:

A nifty demonstration involving increasing series. If you take a constantly increasing series *S*, you can make a new series *M* where the n^{th} term is the number of terms in *S* that are less than *n*. As I don't know the name of this process, I shall refer to *M* as the dual of *S*. Thus:

S	1	2	2	4	5	8	8	8	9
M	0	1	3	3	4	5	5	5	8

If you add the series of counting numbers (1,2,3,4...) to both S and M, all of the counting numbers will be included once in either series. Using the example above:

S_c	2	4	5	8	10	14	15	16	18
M_c	1	3	6	7	9	11	12	13	17

I am assuming that anybody reading this will also have watched the accompanying movie here.

The proof:

The real problem is to find a formula for the non-square numbers. Once we've done that, it's trivial to find the proper index. The method above is perfect: if we can choose a series *S* such that S_c is the set of all squares, M_c will, by definition, contain all the non-squares.

The translation from *S* to S_c is simply adding the index; for the 3rd term, add 3; for the 42nd term, add 42, etc. Therefore, if S_c is the series n^2, then *S* is the series n^2-n.

Now, the real issue is determining M. To assist in my feeble attempts of explanation, I will write out some relevant numbers:

Index	1	2	3	4	5	6	7	8
S [n^2-n]	0	2	6	12	20	30	42	56

Let us suppose we are trying to determine the 7th non-square. The 7th entry in M is equal to the number of terms in S that are less than 7. If you look at the chart, you can see that the answer should be 3: 7 is between 6 and 12, which correspond to 3 and 4. Therefore, the number of terms less than 7 is larger than 3 and less than 4. Since that doesn't make any sense, you can just round it down to 3. Converting this to a quadratic equation ($x^2 - x - n = 0$) and solving with the quadratic formula, we get $\left[\frac{1+\sqrt{1+4n}}{2}\right]=x$. Converting M to M_c, we get $\left[\frac{1+\sqrt{1+4n}}{2}\right]+n=x$.

Let's try it out. Here is a table of some of the values given by the equation:

n	1	2	3	4	5	6	7	8
f(n)	2	4	5	6	7	9	10	11

Oops! Big glitch. Whenever $\sqrt{1+4n}$ is an integer, the returned value is 1 too high. As far as I can tell, my equation is assuming a "less than or equal to" instead of a "less than". The only way I can see to fix this is if the square root is an integer, subtract 1.

To recap:

My completed formula is $\left[\frac{1+\sqrt{1+4n}}{2}\right]+n$. If $\sqrt{1+4n}$ is an integer, subtract 1 before continuing. The square brackets mean "round down". It's not as elegant as I would like, but it seems to work.