Monday Math Madness #7 Submission
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Problem

What is the sum of this infinite series?

$$\frac{F_0}{1} + \frac{3F_1}{5} + \frac{3^2 F_2}{5^2} + \cdots + \frac{3^n F_n}{5^n} + \cdots$$

The subscripted F terms refer to the familiar Fibonacci series where $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for $n > 1$.

Solution #1

The Fibonacci numbers have the well-known generating function

$$F(z) = \sum_{n \geq 0} F_n z^n = \frac{1}{1 - z - z^2}. \tag{1}$$

Setting $z = 3/5$ yields the desired sum right away:

$$F(3/5) = \sum_{n \geq 0} F_n (3/5)^n = \frac{1}{1 - 3/5 - (3/5)^2} = \frac{25}{25 - 15z - 9z} = 25.$$

Solution #2

Denote the given sum by S:

$$S = \sum_{n \geq 0} (3/5)^n F_n. \tag{2}$$

Then we can compute as follows:

$$
\begin{aligned}
(3/5)S + (3/5)^2 S &= \sum_{n\geq 0}(3/5)^{n+1}F_n + \sum_{n\geq 0}(3/5)^{n+2}F_n \\
&= \sum_{n\geq 1}(3/5)^n F_{n-1} + \sum_{n\geq 2}(3/5)^n F_{n-2} \\
&= (3/5)^1 F_0 + \sum_{n\geq 2}(3/5)^n (F_{n-1} + F_{n-2}) \\
&= 3/5 + \sum_{n\geq 2}(3/5)^n F_n \\
&= 3/5 + (S - F_0 - (3/5)F_1) \\
&= S - 1
\end{aligned}
$$

Solving for S, we obtain

$$
S = \frac{1}{1 - 3/5 - (3/5)^2},
$$

which looks surprisingly familiar... in fact, we can use the same method to derive the generating function for the Fibonacci numbers in the first place, simply replacing 3/5 by z throughout. So in a sense, solutions #1 and #2 are really the same solution after all!

Let's do a quick spot-check of our answer with some Haskell code:

```
$ ghci
> let fibs = 1 : 1 : zipWith (+) fibs (tail fibs)
> let pows = iterate (* (3/5)) 1
> take 10 fibs
[1,1,2,3,5,8,13,21,34,55]
> take 4 pows
[1.0,0.6,0.36,0.216]
> sum $ take 500 $ zipWith (*) fibs pows
24.99999079829044
```

Looks good!