Exceedingly Elementary Proofs That $a^{1/n} \to 1$, $n^{1/n} \to 1$, and $(1+1/n)^n \to e$

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1. Introduction

The results in the title are usually proved using calculus. Our goal is to prove them in as elementary a way as possible, without calculus, limits, or functions such as log and exp. It turns out that all that is needed is high-school algebra and induction.

First, some definitions. An elementary proof is one that uses only ordinary (high-school) algebra and induction. No limits, calculus, or functions such as log or exponential are allowed. "Exceedingly elementary" (EE) proofs are elementary proofs in which all variables used are explicitly listed, so that sets of numbers where the number of numbers in the set can be arbitrary, can not be used. As examples, the binomial theorem, arbitrary polynomials, and the Arithmetic-Geometric mean inequality (AGMI) cannot be used in an EE proof. If we are discussing $a^{1/n}$ or $n^{1/n}$, we obviously have to allow taking the n-th root of any positive real, where n is any positive integer.

The goal here is to provide EE proofs of the following results (in all these, n is a positive integer that can grow large, with "→" meaning "in the limit as n gets large"):

1. If a is a positive real number, then $a^{1/n} \to 1$.

2. $n^{1/n} \to 1$.

3. $(1+1/n)^n \to e$.

Of course, none of these are EE, or even elementary, results. What we will actually prove are the following:

1. If $0 < a < 1$ then $a^{1/n} \geq 1 - (1-a)/(na+1-a)$; if $a > 1$, then $a^{1/n} \leq 1+(a-1)/n$.
2. $1+1/n < n^{1/n} < 1 + 3n^{-1/2}$; for any integer $k > 1$, there is an explicit computable constants $c(k)$ and $N(k)$ such that $n^{1/n} < 1 + c(k)n^{-1+1/k}$ whenever $n > N(k)$ – we can take $c(k) = 2k$ when $n>k^{k/(k-1)}$ or, for any $c > 0$, we can show $n^{1/n} < 1+(1+c)kn^{-1+1/k}$ if $n > ((1+c)k/c)^{k/(k-1)}$.
3. $(1+1/n)^n$ is strictly increasing, $(1+1/n)^{n+1}$ is strictly decreasing, and their difference can be made arbitrarily small (explicitly, $0 < (1+1/n)^{n+1} - (1+1/n)^n < 3/n$).

The only parts of these that are original with the author are the second part of (2) and the proving of (3) by EE means. (1) and (essentially) the first part of (2) are in [1], page 324; (3) is proved using the the AGMI in [2] (pointed to by a sci.math posting).

All details of all the proofs are explicitly given so that the claim of being EE can be easily verified.

The primary result used in proving these is Bernoulli's inequality (BI):

If n is a positive integer, and $x \geq 0$ then $(1+x)^n \geq 1+nx$; if, in addition, $n \geq 2$ and $x > 0$, the inequality is strict.

The proofs of both parts are by induction: For the first part, it is true for $n = 1$; if it is true for n, then $(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1+(n+1)x+nx^2 \geq 1+(n+1)x$.
For the second part ($x \neq 0$), it is true for $n=2$ since $(1+x)^2 = 1+2x+x^2 > 1+2x$; if it is true for n then $(1+x)^{n+1} = (1+x)(1+x)^n > (1+x)(1+nx) = 1+(n+1)x+nx^2 > 1+(n+1)x$.

2. Proof that $a^{1/n} \to 1$

These are modifications of the proof in [1] and use BI.

If $a > 1$, let $a^{1/n} = 1+b$. Then $a = (1+b)^n \geq 1+nb$, so $b \leq (a-1)/n$, or $a^{1/n} \leq 1+(a-1)/n$.

If $0 < a < 1$, let $a^{1/n} = 1/(1+b)$. Then $1/a = (1+b)^n \geq 1+bn$, so $b \leq (1/a-1)/n$ and $a^{1/n} \geq 1/(1+(1/a-1)/n) = 1 - ((1/a-1)/n)/(1+(1/a-1)/n) = 1 - (1-a)/(na+1-a)$.

In both cases, by choosing n large enough, we can make $a^{1/n}$ as close to 1 as we want. Explicitly, for any $c > 0$, we can explicitly choose n to make $a^{1/n}$ within c of 1 by making $n > (a-1)/c$ in the first case and $n > ((1-a)/c-1+a)/a$ in the second case.

3. Proofs that $n^{1/n} \to 1$

My first proof that $n^{1/n} \to 1$ came when I noticed that $(1+c)^n \geq 1+cn > cn$, so that $1+c > c^{1/n}n^{1/n}$. Since both 1+c and $c^{1/n}$ approach 1 (first choosing c small, then choosing n large), $n^{1/n}$ also approaches 1. The rest is in the details.

The first explicit version of the proof used this directly. We have, if $0 < c < 1$, $(1/c)^{1/n} < 1+(1/c-1)/n < 1+1/cn$, so that $n^{1/n} < (1+c)(1/c)^{1/n} < (1+c)(1+1/(cn))$. If $n > 1/c^2$, $n^{1/n} < (1+c)^2 = 1+2c+c^2 < 1+3c$ so that, substituting c/3 for c, if $n > 9/c^2$ then $n^{1/n} < 1+c$ for any $c > 0$.

For the next version, I removed the variable "c" from the proof and worked directly with n. By setting $c = n^{-1/2}$, I came up with this shorty: $(1+n^{-1/2})^n \geq 1+n^{1/2} > n^{1/2}$. Taking the n-th root, $n^{1/2n} < 1+n^{-1/2}$, or $n^{1/n} < (1+n^{-1/2})^2 = 1 + 2n^{-1/2} + n^{-1} < 1 + 3n^{-1/2}$. (This proof is very similar to the proof in [1] that $n^{1/n} \to 1$.)

For the final version, I wanted to see how much the exponent of n could be improved. Since the true value is $n^{1/n} = e^{\ln(n)/n} \sim 1 + \ln(n)/n$ for large n, and $\ln(n) \sim k(n^{1/k}-1)$ for large k (since $n^{1/k} = e^{\ln(n)/k} \sim 1+\ln(n)/k$), $n^{1/n} \sim 1+kn^{1/k}/n$ for large n and k. My goal was to show that $n^{1/n} < 1+c(k)n^{1/k}/n$ for every integral $k \geq 2$ and some c(k) for large enough n (the previous result showed this for k = 2 with c(2)=3). To do this, after some experimentation with the exponent of n (in $(1+n^a)^n$), I found this: $(1+n^{-1+1/k})^n \geq 1+n^{1/k} > n^{1/k}$, so, raising both sides to the k/n power, $n^{1/n} < (1+n^{-1+1/k})^k$. What is needed is upper bounds for $(1+n^{-1+1/k})^k$. The first 2 terms of the binomial theorem ($(1+n^{-1+1/k})^k \sim 1+kn^{-1+1/k} + \ldots$) (as well as the result, above, that $n^{1/n} \sim 1+kn^{1/k}/n$ for large n and k) give an idea of the best we can do. It turns out that we can get, by EE means, arbitrarily close, which I found pleasantly surprising.

What is needed here is an upper bound to $(1+x)^k$ for $x > 0$ and integral $k \geq 2$. There cannot be a bound of the form $(1+x)^k \leq 1+c(k)x$ for all x, because $(1+x)^k > 1+x^k$ so that, if $(1+x)^k \leq 1+c(k)x$, then $x < (c(k))^{1/(k-1)}$. Thus, we must assume that x is bounded.

I have come up with two of these contra-Bernoulli inequalities, or CBIs (as I call them). They are (k is a positive integer):

1. if $0 < x < 1/k$, $(1+x)^k < 1/(1-kx)$;
2. if $0 < x < M$, then $(1+x)^k \leq 1+c(k,M)x$, where $c(k,M) = ((1+M)^k-1)/M$.

As usual, we prove them by induction.

We prove CBI 1 in the form $(1+x)^k(1-kx) < 1$. It is true for k=1, since $(1+x)(1-x) = 1-x^2 < 1$. If it is true for k, then we want to show that $(1+x)^{k+1}(1-(k+1)x) < 1$ if $0 < x < 1/(k+1)$. But $(1+x)^{k+1}(1-(k+1)x) = (1+x)(1+x)^k(1-(k+1)x)$ $< (1+x)(1-(k+1)x)/(1-kx) = (1-kx-(k+1)x^2)/(1-kx) < (1-kx)/(1-kx) = 1$.

The proof of CBI 2 is in two parts. We first show that if $c(1,M)=1$ and $c(k+1,M) = 1+(M+1)c(k,M)$, then $(1+x)^n \leq 1+c(k,M)x$. We then show that $c(k,M) = ((1+M)^k-1)/M$. Both proofs are by induction.

The first part is true for k=1 ($1+x \le 1+x$). If it is true for k, then

$(1+x)^{k+1} = (1+x)(1+x)^k < (1+x)(1+c(k,M)x) = 1+x(1+c(k,M))+c(k,M)x^2$

$< 1+x(1+c(k,M))+c(k,M)Mx$ (this is where we use $0 < x < M$)

$= 1 + x(1+c(k,M)(M+1)) = 1+c(k+1,M)x$.

The second part is true for k=1. If it is true for k, since $c(k.M) = ((M+1)^k-1)/M$, then

$(1+c(k,M)(M+1)) = (1+ (M+1)((M+1)^k-1)/M) = ((M+1)^{k+1}-M-1+M)/M = ((M+1)^{k+1}-1)/M$.

A second proof of CBI 2 is less rigorous, but provides a easy to see "reason" why it is true. It depends on the coefficients of $(1+x)^k$ being all positive with constant term 1. This can be proved by induction if it is not "obvious". (This can also lead to a discussion of the binomial theorem.) Then $((1+x)^k-1)/x$ is a polynomial with all coefficients positive, so this is an increasing function with maximum value for x from 0 to M of $((1+M)^k-1)/M$. This proof is not, however, EE, and requires a surprisingly large amount of work to be made rigorous.

CBI 1 gives us $n^{1/n} < (1+n^{-1+1/k})^k < 1/(1-kn^{-1+1/k})$ if $n > k^{k/(k-1)}$ (from $n^{-1+1/k} < 1/k$). To convert this to an inequality of the form $n^{1/n} < 1+c(k)n^{-1+1/k}$, we use the result that, if x and c are positive, $1/(1-x) < 1+(1+c)x$ for $x < c/(1+c)$

(Proof: $1/(1-x) < (1+(1+c)x) \Leftrightarrow 1+cx-(1+c)x^2 > 1 \Leftrightarrow cx > (1+c)x^2 \Leftrightarrow x < c/(1+c)$),

so that $n^{1/n} < 1+(1+c)kn^{-1+1/k}$ if $n > ((1+c)k/c)^{k/(k-1)}$ (from $kn^{1+1/k} < c/(1+c)$). If c=1, this gives $n^{1/n} < 1+2kn^{-1+1/k}$ if $n > (2k)^{k/(k-1)}$.

CBI 2 gives us this: if $n^{-1+1/k} \le M$, then $n^{1/n} < 1+c(k,M)n^{-1+1/k} = 1+((M+1)^k-1)/M)n^{-1+1/k}$. We now see what values of M can be used for various values of n and $k > 1$.

Simplest (but worst), since $0 < n^{1/k-1} < 1$, $n^{1/n} < 1+c(k,1)n^{1/k-1} = 1+(2^k-1)n^{1/k-1}$ for all n and $k > 1$. This is the result above with $k = 2$.

Next, if $n^{-1+1/k} \le 1/k$, or $n > k^{k/(k-1)}$, $M = 1/k$, so $c(k,M) = ((1+1/k)^k-1)/(1/k) = k((1+1/k)^k-1)$. If we use the result $(1+1/k)^k < 3$ (which is proved in the next section), we get $n^{1/n} < 1+2kn^{-1+1/k}$ for $n>k^{k/(k-1)}$. Note that this is a little better that the result using CBI 1, above.

Finally, we can use CBI 1 in this use of CBI 2 to get an upper bound on c(k,M). If $n^{-1+1/k} < 1/k^j$, where j is an integer greater than 1 (equivalently, $n > k^{jk/(k-1)}$), we can choose $M = 1/k^j$, so $(1+M)^k < 1/(1-kM) = 1/(1-k^{-j+1})$, so

$c(k,M) < (1/(1-k^{-j+1})-1)k^j = k/ (1-k^{-j+1}) = k(1+1/(k^{j-1}-1))$.

For example, if j=2, this shows that if $k > 1$ and $n > k^{2k/(k-1)}$, then

$n^{1/n} < 1+k(1+1/(k-1))n^{-1+1/k}$. This shows that, by choosing n large enough, we can get quite close to the optimum result $1+kn^{-1+1/k}$.

Note that both of these results show that, for any $c > 0$, if n is large enough compared to k, then $n^{1/n} < 1 + (1+c)kn^{-1+1/k}$, which is impressively (to me) close to the optimal result shown above that $n^{1/n} \sim 1 + kn^{-1+1/k}$.

As a final note, we prove the lower bound $n^{1/n} > 1+1/n$ for $n \geq 3$. This follows from the result proved in the next section that $(1+1/n)^n < 3$, but can also be proved directly. This lower bound is equivalent to $(n+1)^n/n^{(n+1)} < 1$. Interestingly, I found it easier to prove that the sequence $(n+1)^n/n^{(n+1)}$ is less than 1 for n=3 ($4^3/3^4=64/81<1$) and decreasing for $n \geq 3$ than to directly prove the inequality (even though the inequality is equivalent to $(1+1/n)^n < n$, which is far from the best possible). To show the sequence is decreasing, $(n+1)^n/n^{(n+1)} > (n+2)^{(n+1)}/(n+1)^{(n+2)} \Leftrightarrow (n+1)^{2n+2} > (n(n+2))^{n+1} \Leftrightarrow (n^2+2n+1)^{n+1} > (n^2+2n)^{n+1}$, which is pretty clear.

4. Proof that $(1+1/n)^n \to e$

Let $a_n = (1+1/n)^n$ and $b_n = (1+1/n)^{n+1}$. We will prove that a_n is an increasing sequence and b_n is an decreasing sequence. Since $a_n < b_n$, this implies, for any positive integers n and m with $m < n$ that $a_m < a_n < b_n < b_m$.

We use the very ingenious proof in [2], which uses the AGMI, which we will use in the form $((v_1+v_2+...v_n)/n)^n \geq v_1v_2...v_n$ (all v_i positive) with equality if and only if all the v_i are equal (this allows us to avoid the use of n-th roots). For a_n, consider n values of $1+1/n$ and 1 value of 1. By the AGMI, $((n+2)/(n+1))^{n+1} > (1+1/n)^n$, or $(1+1/(n+1))^{n+1} > (1+1/n)^n$, or $a_{n+1} > a_n$. For b_n, consider n values of $1-1/n$ and 1 value of 1. By the AGMI, $(n/(n+1))^{n+1} > (1-1/n)^n$ or $(1+1/n)^{n+1} < (1+1/(n-1))^n$, or $b_n < b_{n+1}$.

These proofs do not seem to be EE, since they use the AGMI. However, they use a special form of the AGMI, where all but one of the values are the same, and this will now be shown to be implied by BI and thus is EE.

Suppose we have n-1 values of u and 1 value of v with u and v positive. The AGMI for these values is

$(((n-1)u+v)/n)^n \geq u^{n-1}v$ with equality if and only if u = v.

We will now show that this is implied by BI:

$(((n-1)u+v)/n)^n \geq u^{n-1}v$ is the same as $(u+(v-u)/n)^n \geq u^n(v/u)$. Dividing by u^n, this is equivalent to $(1+(v/u-1)/n)^n \geq v/u$. By BI, $(1+(v/u-1)/n)^n \geq 1+n((v/u-1)/n) = v/u$ with equality only if n=1 or v/u-1 = 0. Since BI is EE, so is this version of the AGMI.

Since the a_n are increasing, the b_n are decreasing, and $a_n < b_n$, all the a_n are less than any of the b_n. Since $b_5 = 2.9859... < 3$, all the a_n are less than 3. If n > m, since $a_n < b_m$ and $b_m-a_m = a_m/m$, $a_n-a_m < b_m-a_m = a_m/m < 3/m$. Once the Cauchy criterion for convergence of a series has been introduced, it can be used to show that the a_n and b_n converge to a common limit, which, since it was proved by EE means, I propose we call "e".

1. R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, New York, 1941.

2. N.S Mendelsohn, An application of a famous inequality, Amer. Math. Monthly 58 (1951), 563.